Distributions on symmetric cones II: Beta-Riesz distribution

José A. Díaz-García *
Department of Statistics and Computation
25350 Buenavista, Saltillo, Coahuila, Mexico
E-mail: jadiaz@uaaan.mx

Abstract
This article derives several properties of the Riesz distributions, such as their corresponding Bartlett decompositions, the inverse Riesz distributions and the distribution of the generalised variance for real normed division algebras. In addition, introduce a kind of generalised beta distribution termed beta-Riesz distribution for real normed division algebras. Two versions of this distributions are proposed and some properties are studied.

1 Introduction

It is imminent the important role played by Wishart and beta distributions type I and II in the context of multivariate statistics. In particular, the relationship between these two distributions to obtain the beta distribution in terms of the distribution of two Wishart matrices. Faraut and Korányi (1994), and subsequently Hassairi *et al.* (2005), propose a beta-Riesz distribution, which contains as a special case to the beta distribution obtained in terms of the distribution Wishart, which shall be named beta-Wishart, all this subjects in the context of simple Euclidean Jordan algebras. Such beta-Riesz distribution is obtained analogously to the beta-Wishart distribution, but starting with a Riesz distribution. Recently, Díaz-García (2012) proposes two versions of the Riesz distribution for real normed division algebras.

Based in these last two versions of the Riesz distributions, it is possible to obtain two versions of the beta-Riesz distributions, which by analogy with the beta-Wishart distributions are termed beta-Riesz type I. As in classical beta-Wishart distribution, in addition it is feasible to propose two version for the beta-Riesz distribution type II. Each of the two versions for each beta-Riesz distributions of type I and II, (both versions for each) contain as particular cases to beta-Wishart distribution type I and beta-Wishart distribution type II, respectively.

This article studies two versions for beta-Riesz distributions type I and II for real normed division algebras. Section 2 reviews some definitions and notation on real normed division algebras. And also, introduces other mathematical tools as two definitions of the generalised gamma function on symmetric cones, two Jacobians with respect to Lebesgue measure and some integral results for real normed division algebras. Section 3 proposes diverse properties

*Corresponding author
Key words. Wishart distribution; Beta-Riesz distribution; Riesz distribution, generalised beta function and distribution, real, complex quaternion and octonion random matrices.
2000 Mathematical Subject Classification. Primary 60E05, 62E15; secondary 15A52

of two versions of the Riesz distributions as their Bartlett decompositions, inverse Riesz distributions and the distribution of the generalized variance. Section 4 introduces two generalised beta functions and, in terms of these, two beta-Riesz distributions of type I and II are obtained for real normed division algebras. Also, the relationship between the Riesz distributions and the beta-Riesz distributions are studied. This section concludes studying the eigenvalues distributions of beta-Riesz distributions type I and II in their two versions for real normed division algebras.

2 Preliminary results

A detailed discussion of real normed division algebras may be found in Baez (2002) and Ebbinghaus *et al.* (1990). For convenience, we shall introduce some notation, although in general we adhere to standard notation forms.

For our purposes: Let $\mathbb{F}$ be a field. An *algebra* $\mathfrak{A}$ over $\mathbb{F}$ is a pair $(\mathfrak{A}; m)$, where $\mathfrak{A}$ is a *finite-dimensional vector space* over $\mathbb{F}$ and *multiplication* $m : \mathfrak{A} \times \mathfrak{A} \to A$ is an $\mathbb{F}$-bilinear map; that is, for all $\lambda \in \mathbb{F}$, $x, y, z \in \mathfrak{A}$,

$$\begin{aligned} m(x, \lambda y + z) &= \lambda m(x; y) + m(x; z) \\ m(\lambda x + y; z) &= \lambda m(x; z) + m(y; z). \end{aligned}$$

Two algebras $(\mathfrak{A}; m)$ and $(\mathfrak{E}; n)$ over $\mathbb{F}$ are said to be *isomorphic* if there is an invertible map $\phi : \mathfrak{A} \to \mathfrak{E}$ such that for all $x, y \in \mathfrak{A}$,

$$\phi(m(x, y)) = n(\phi(x), \phi(y)).$$

By simplicity, we write $m(x; y) = xy$ for all $x, y \in \mathfrak{A}$.

Let $\mathfrak{A}$ be an algebra over $\mathbb{F}$. Then $\mathfrak{A}$ is said to be

1. *alternative* if $x(xy) = (xx)y$ and $x(yy) = (xy)y$ for all $x, y \in \mathfrak{A}$,
2. *associative* if $x(yz) = (xy)z$ for all $x, y, z \in \mathfrak{A}$,
3. *commutative* if $xy = yx$ for all $x, y \in \mathfrak{A}$, and
4. *unital* if there is a $1 \in \mathfrak{A}$ such that $x1 = x = 1x$ for all $x \in \mathfrak{A}$.

If $\mathfrak{A}$ is unital, then the identity 1 is uniquely determined.

An algebra $\mathfrak{A}$ over $\mathbb{F}$ is said to be a *division algebra* if $\mathfrak{A}$ is nonzero and $xy = 0_{\mathfrak{A}} \Rightarrow x = 0_{\mathfrak{A}}$ or $y = 0_{\mathfrak{A}}$ for all $x, y \in \mathfrak{A}$.

The term "division algebra", comes from the following proposition, which shows that, in such an algebra, left and right division can be unambiguously performed.

Let $\mathfrak{A}$ be an algebra over $\mathbb{F}$. Then $\mathfrak{A}$ is a division algebra if, and only if, $\mathfrak{A}$ is nonzero and for all $a, b \in \mathfrak{A}$, with $b \neq 0_{\mathfrak{A}}$, the equations $bx = a$ and $yb = a$ have unique solutions $x, y \in \mathfrak{A}$.

In the sequel we assume $\mathbb{F} = \Re$ and consider classes of division algebras over $\Re$ or "*real division algebras*" for short.

We introduce the algebras of *real numbers* $\Re$, *complex numbers* $\mathfrak{C}$, *quaternions* $\mathfrak{H}$ and *octonions* $\mathfrak{O}$. Then, if $\mathfrak{A}$ is an alternative real division algebra, then $\mathfrak{A}$ is isomorphic to $\Re$, $\mathfrak{C}$, $\mathfrak{H}$ or $\mathfrak{O}$.

Let $\mathfrak{A}$ be a real division algebra with identity 1. Then $\mathfrak{A}$ is said to be *normed* if there is an inner product $(\cdot, \cdot)$ on $\mathfrak{A}$ such that

$$(xy, xy) = (x, x)(y, y) \qquad \text{for all } x, y \in \mathfrak{A}.$$

If $\mathfrak{A}$ is a *real normed division algebra*, then $\mathfrak{A}$ is isomorphic $\Re$, $\mathfrak{C}$, $\mathfrak{H}$ or $\mathfrak{O}$.

There are exactly four normed division algebras: real numbers ($\Re$), complex numbers ($\mathfrak{C}$), quaternions ($\mathfrak{H}$) and octonions ($\mathfrak{O}$), see Baez (2002). We take into account that should be taken into account, $\Re$, $\mathfrak{C}$, $\mathfrak{H}$ and $\mathfrak{O}$ are the only normed division algebras; furthermore, they are the only alternative division algebras.

Let $\mathfrak{A}$ be a division algebra over the real numbers. Then $\mathfrak{A}$ has dimension either 1, 2, 4 or 8. In other branches of mathematics, the parameters $\alpha = 2/\beta$ and $t = \beta/4$ are used, see Edelman and Rao (2005) and Kabe (1984), respectively.

Finally, observe that

$\Re$ is a real commutative associative normed division algebras,

$\mathfrak{C}$ is a commutative associative normed division algebras,

$\mathfrak{H}$ is an associative normed division algebras,

$\mathfrak{O}$ is an alternative normed division algebras.

Let $\mathcal{L}^{\beta}_{m,n}$ be the set of all $m \times n$ matrices of rank $m \leq n$ over $\mathfrak{A}$ with m distinct positive singular values, where $\mathfrak{A}$ denotes a *real finite-dimensional normed division algebra*. Let $\mathfrak{A}^{m\times n}$ be the set of all $m \times n$ matrices over $\mathfrak{A}$. The dimension of $\mathfrak{A}^{m\times n}$ over $\Re$ is βmn. Let $\mathbf{A} \in \mathfrak{A}^{m\times n}$, then $\mathbf{A}^* = \bar{\mathbf{A}}^T$ denotes the usual conjugate transpose.

Table 1 sets out the equivalence between the same concepts in the four normed division algebras.

Table 1: Notation

Real	Complex	Quaternion	Octonion	Generic notation
Semi-orthogonal	Semi-unitary	Semi-symplectic	Semi-exceptional type	$\mathcal{V}^{\beta}_{m,n}$
Orthogonal	Unitary	Symplectic	Exceptional type	$\mathfrak{U}^{\beta}(m)$
Symmetric	Hermitian	Quaternion hermitian	Octonion hermitian	$\mathfrak{S}^{\beta}_{m}$

We denote by $\mathfrak{S}^{\beta}_{m}$ the real vector space of all $\mathbf{S} \in \mathfrak{A}^{m\times m}$ such that $\mathbf{S} = \mathbf{S}^*$. In addition, let $\mathfrak{P}^{\beta}_{m}$ be the *cone of positive definite matrices* $\mathbf{S} \in \mathfrak{A}^{m\times m}$. Thus, $\mathfrak{P}^{\beta}_{m}$ consist of all matrices $\mathbf{S} = \mathbf{X}^*\mathbf{X}$, with $\mathbf{X} \in \mathfrak{L}^{\beta}_{m,n}$; then $\mathfrak{P}^{\beta}_{m}$ is an open subset of $\mathfrak{S}^{\beta}_{m}$.

Let $\mathfrak{D}^{\beta}_{m}$ be the *diagonal subgroup* of $\mathcal{L}^{\beta}_{m,m}$ consisting of all $\mathbf{D} \in \mathfrak{A}^{m\times m}$, $\mathbf{D} = \operatorname{diag}(d_1, \dots, d_m)$. Let $\mathfrak{T}^{\beta}_{U}(m)$ be the subgroup of all *upper triangular* matrices $\mathbf{T} \in \mathfrak{A}^{m\times m}$ such that $t_{ij} = 0$ for $1 < i < j \leq m$.

For any matrix $\mathbf{X} \in \mathfrak{A}^{n\times m}$, $d\mathbf{X}$ denotes the *matrix of differentials* (dx_{ij}). Finally, we define the *measure* or volume element $(d\mathbf{X})$ when $\mathbf{X} \in \mathfrak{A}^{m\times n}$, $\mathfrak{S}^{\beta}_{m}$, $\mathfrak{D}^{\beta}_{m}$ or $\mathcal{V}^{\beta}_{m,n}$, see Dumitriu (2002) and Díaz-García and Gutiérrez-Jáimez (2011).

If $\mathbf{X} \in \mathfrak{A}^{m\times n}$ then $(d\mathbf{X})$ (the Lebesgue measure in $\mathfrak{A}^{m\times n}$) denotes the exterior product of the βmn functionally independent variables

$$(d\mathbf{X}) = \bigwedge_{i=1}^{m}\bigwedge_{j=1}^{n} dx_{ij} \quad \text{where} \quad dx_{ij} = \bigwedge_{k=1}^{\beta} dx_{ij}^{(k)}.$$

If $\mathbf{S} \in \mathfrak{S}^{\beta}_{m}$ (or $\mathbf{S} \in \mathfrak{T}^{\beta}_{U}(m)$ with $t_{ii} > 0$, $i = 1, \dots, m$) then $(d\mathbf{S})$ (the Lebesgue measure in $\mathfrak{S}^{\beta}_{m}$ or in $\mathfrak{T}^{\beta}_{U}(m)$) denotes the exterior product of the exterior product of the $m(m-1)\beta/2+m$ functionally independent variables,

$$(d\mathbf{S}) = \bigwedge_{i=1}^{m} ds_{ii} \bigwedge_{i<j}^{m}\bigwedge_{k=1}^{\beta} ds_{ij}^{(k)}.$$

Observe, that for the Lebesgue measure $(d\mathbf{S})$ defined thus, it is required that $\mathbf{S} \in \mathfrak{P}_m^\beta$, that is, $\mathbf{S}$ must be a non singular Hermitian matrix (Hermitian definite positive matrix).

If $\mathbf{\Lambda} \in \mathfrak{D}_m^\beta$ then $(d\mathbf{\Lambda})$ (the Legesgue measure in $\mathfrak{D}_m^\beta$) denotes the exterior product of the βm functionally independent variables

$$(d\mathbf{\Lambda}) = \bigwedge_{i=1}^{n} \bigwedge_{k=1}^{\beta} d\lambda_i^{(k)}.$$

If $\mathbf{H}_1 \in \mathcal{V}_{m,n}^\beta$ then

$$(\mathbf{H}_1^* d\mathbf{H}_1) = \bigwedge_{i=1}^{m} \bigwedge_{j=i+1}^{n} \mathbf{h}_j^* d\mathbf{h}_i.$$

where $\mathbf{H} = (\mathbf{H}_1^*|\mathbf{H}_2^*)^* = (\mathbf{h}_1, \dots, \mathbf{h}_m|\mathbf{h}_{m+1}, \dots, \mathbf{h}_n)^* \in \mathfrak{U}^\beta(n)$. It can be proved that this differential form does not depend on the choice of the $\mathbf{H}_2$ matrix. When $n = 1$; $\mathcal{V}_{m,1}^\beta$ defines the unit sphere in $\mathfrak{A}^m$. This is, of course, an $(m-1)\beta$- dimensional surface in $\mathfrak{A}^m$. When $n = m$ and denoting $\mathbf{H}_1$ by $\mathbf{H}$, $(\mathbf{H}d\mathbf{H}^*)$ is termed the *Haar measure* on $\mathfrak{U}^\beta(m)$.

The surface area or volume of the Stiefel manifold $\mathcal{V}_{m,n}^\beta$ is

$$\operatorname{Vol}(\mathcal{V}_{m,n}^\beta) = \int_{\mathbf{H}_1 \in \mathcal{V}_{m,n}^\beta} (\mathbf{H}_1 d\mathbf{H}_1^*) = \frac{2^m \pi^{mn\beta/2}}{\Gamma_m^\beta[n\beta/2]}, \tag{1}$$

where $\Gamma_m^\beta[a]$ denotes the multivariate *Gamma function* for the space $\mathfrak{S}_m^\beta$. This can be obtained as a particular case of the *generalised gamma function of weight* κ for the space $\mathfrak{S}_m^\beta$ with $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \dots \geq k_m \geq 0$, taking $\kappa = (0, 0, \dots, 0)$ and which for $\operatorname{Re}(a) \geq (m-1)\beta/2 - k_m$ is defined by, see Gross and Richards (1987),

$$\begin{aligned}
\Gamma_m^\beta[a, \kappa] &= \int_{\mathbf{A} \in \mathfrak{P}_m^\beta} \operatorname{etr}\{-\mathbf{A}\} |\mathbf{A}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{A}) (d\mathbf{A}) && (2)\\
&= \pi^{m(m-1)\beta/4} \prod_{i=1}^{m} \Gamma[a + k_i - (i-1)\beta/2] \\
&= [a]_\kappa^\beta \Gamma_m^\beta[a], && (3)
\end{aligned}$$

where $\operatorname{etr}(\cdot) = \exp(\operatorname{tr}(\cdot))$, $|\cdot|$ denotes the determinant, and for $\mathbf{A} \in \mathfrak{S}_m^\beta$

$$q_\kappa(\mathbf{A}) = |\mathbf{A}_m|^{k_m} \prod_{i=1}^{m-1} |\mathbf{A}_i|^{k_i - k_{i+1}} \tag{4}$$

with $\mathbf{A}_p = (a_{rs})$, $r, s = 1, 2, \dots, p$, $p = 1, 2, \dots, m$ is termed the *highest weight vector*, see Gross and Richards (1987). Also,

$$\begin{aligned}
\Gamma_m^\beta[a] &= \int_{\mathbf{A} \in \mathfrak{P}_m^\beta} \operatorname{etr}\{-\mathbf{A}\} |\mathbf{A}|^{a-(m-1)\beta/2-1} (d\mathbf{A}) \\
&= \pi^{m(m-1)\beta/4} \prod_{i=1}^{m} \Gamma[a - (i-1)\beta/2],
\end{aligned}$$

and $\operatorname{Re}(a) > (m-1)\beta/2$.

In other branches of mathematics the *highest weight vector* $q_\kappa(\mathbf{A})$ is also termed the *generalised power* of $\mathbf{A}$ and is denoted as $\Delta_\kappa(\mathbf{A})$, see Faraut and Korányi (1994) and Hassairi and Lajmi (2001).

Additional properties of $q_\kappa(\mathbf{A})$, which are immediate consequences of the definition of $q_\kappa(\mathbf{A})$ and the following property 1, are:

1. if $\lambda_1, \dots, \lambda_m$, are the eigenvalues of $\mathbf{A}$, then
$$q_\kappa(\mathbf{A}) = \prod_{i=1}^{m} \lambda_i^{k_i}. \tag{5}$$

2. $$q_\kappa(\mathbf{A}^{-1}) = q_\kappa^{-1}(\mathbf{A}) = q_{-\kappa}(\mathbf{A}), \tag{6}$$

3. if $\kappa = (p, \dots, p)$, then
$$q_\kappa(\mathbf{A}) = |\mathbf{A}|^p, \tag{7}$$
in particular if $p = 0$, then $q_\kappa(\mathbf{A}) = 1$.

4. if $\tau = (t_1, t_2, \dots, t_m)$, $t_1 \geq t_2 \geq \cdots \geq t_m \geq 0$, then
$$q_{\kappa+\tau}(\mathbf{A}) = q_\kappa(\mathbf{A}) q_\tau(\mathbf{A}), \tag{8}$$
in particular if $\tau = (p, p, \dots, p)$, then
$$q_{\kappa+\tau}(\mathbf{A}) \equiv q_{\kappa+p}(\mathbf{A}) = |\mathbf{A}|^p q_\kappa(\mathbf{A}). \tag{9}$$

5. Finally, for $\mathbf{B} \in \mathfrak{A}^{m \times m}$ in such a manner that $\mathbf{C} = \mathbf{B}^*\mathbf{B} \in \mathfrak{S}_m^\beta$,
$$q_\kappa(\mathbf{BAB}^*) = q_\kappa(\mathbf{C}) q_\kappa(\mathbf{A}) \tag{10}$$
and
$$q_\kappa(\mathbf{B}^{-1}\mathbf{AB}^{*-1}) = (q_\kappa(\mathbf{C}))^{-1} q_\kappa(\mathbf{A}). \tag{11}$$

Remark 2.1. Let $\mathcal{P}(\mathfrak{S}_m^\beta)$ denote the algebra of all polynomial functions on $\mathfrak{S}_m^\beta$, and $\mathcal{P}_k(\mathfrak{S}_m^\beta)$ the subspace of homogeneous polynomials of degree k and let $\mathcal{P}^\kappa(\mathfrak{S}_m^\beta)$ be an irreducible subspace of $\mathcal{P}(\mathfrak{S}_m^\beta)$ such that
$$\mathcal{P}_k(\mathfrak{S}_m^\beta) = \sum_\kappa \bigoplus \mathcal{P}^\kappa(\mathfrak{S}_m^\beta).$$

Note that q_κ is a homogeneous polynomial of degree k, moreover $q_\kappa \in \mathcal{P}^\kappa(\mathfrak{S}_m^\beta)$, see Gross and Richards (1987).

In (3), $[a]_\kappa^\beta$ denotes the generalised Pochhammer symbol of weight κ, defined as
$$\begin{aligned}
[a]_\kappa^\beta &= \prod_{i=1}^{m} (a - (i-1)\beta/2)_{k_i} \\
&= \frac{\pi^{m(m-1)\beta/4} \displaystyle\prod_{i=1}^{m} \Gamma[a + k_i - (i-1)\beta/2]}{\Gamma_m^\beta[a]} \\
&= \frac{\Gamma_m^\beta[a, \kappa]}{\Gamma_m^\beta[a]},
\end{aligned}$$

where $\mathrm{Re}(a) > (m-1)\beta/2 - k_m$ and
$$(a)_i = a(a+1)\cdots(a+i-1),$$

is the standard Pochhammer symbol.

An alternative definition of the generalised gamma function of weight κ is proposed by Khatri (1966), which is defined as

$$\begin{aligned}
\Gamma_m^\beta[a,-\kappa] &= \int_{\mathbf{A}\in\mathfrak{P}_m^\beta} \operatorname{etr}\{-\mathbf{A}\}|\mathbf{A}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{A}^{-1})(d\mathbf{A}) \qquad (12)\\
&= \pi^{m(m-1)\beta/4}\prod_{i=1}^m \Gamma[a-k_i-(m-i)\beta/2]\\
&= \frac{(-1)^k\Gamma_m^\beta[a]}{[-a+(m-1)\beta/2+1]_\kappa^\beta}, \qquad (13)
\end{aligned}$$

where $\operatorname{Re}(a) > (m-1)\beta/2+k_1$.

Similarly, from Herz (1955, p. 480) the *multivariate beta function* for the space $\mathfrak{S}_m^\beta$, can be defined as

$$\begin{aligned}
\mathcal{B}_m^\beta[b,a] &= \int_{\mathbf{0}<\mathbf{S}<\mathbf{I}_m} |\mathbf{S}|^{a-(m-1)\beta/2-1}|\mathbf{I}_m-\mathbf{S}|^{b-(m-1)\beta/2-1}(d\mathbf{S})\\
&= \int_{\mathbf{R}\in\mathfrak{P}_m^\beta} |\mathbf{R}|^{a-(m-1)\beta/2-1}|\mathbf{I}_m+\mathbf{R}|^{-(a+b)}(d\mathbf{R})\\
&= \frac{\Gamma_m^\beta[a]\Gamma_m^\beta[b]}{\Gamma_m^\beta[a+b]}, \qquad (14)
\end{aligned}$$

where $\mathbf{R} = (\mathbf{I}-\mathbf{S})^{-1}-\mathbf{I}$, $\operatorname{Re}(a) > (m-1)\beta/2$ and $\operatorname{Re}(b) > (m-1)\beta/2$, see Díaz-García and Gutiérrez-Jáimez (2009b).

Now, we show three Jacobians in terms of the β parameter, which are proposed as extensions of real, complex or quaternion cases, see Díaz-García and Gutiérrez-Jáimez (2011).

Lemma 2.1. *Let $\mathbf{X}$ and $\mathbf{Y} \in \mathfrak{P}_m^\beta$ matrices of functionally independent variables, and let $\mathbf{Y} = \mathbf{AXA}^* + \mathbf{C}$, where $\mathbf{A} \in \mathcal{L}_{m,m}^\beta$ and $\mathbf{C} \in \mathfrak{P}_m^\beta$ are matrices of constants. Then*

$$(d\mathbf{Y}) = |\mathbf{A}^*\mathbf{A}|^{\beta(m-1)/2+1}(d\mathbf{X}). \qquad (15)$$

Lemma 2.2 (Cholesky's decomposition)**.** *Let $\mathbf{S} \in \mathfrak{P}_m^\beta$ and $\mathbf{T} \in \mathfrak{T}_U^\beta(m)$ with $t_{ii} > 0$, $i = 1, 2, \dots, m$. Then*

$$(d\mathbf{S}) = \begin{cases} 2^m \displaystyle\prod_{i=1}^m t_{ii}^{\beta(m-i)+1}(d\mathbf{T}) & \text{if } \mathbf{S} = \mathbf{T}^*\mathbf{T};\\ 2^m \displaystyle\prod_{i=1}^m t_{ii}^{\beta(i-1)+1}(d\mathbf{T}) & \text{if } \mathbf{S} = \mathbf{T}\mathbf{T}^*. \end{cases} \qquad (16)$$

Lemma 2.3. *Let $\mathbf{S} \in \mathfrak{P}_m^\beta$. Then, ignoring the sign, if $\mathbf{Y} = \mathbf{S}^{-1}$*

$$(d\mathbf{Y}) = |\mathbf{S}|^{-\beta(m-1)-2}(d\mathbf{S}). \qquad (17)$$

Next is stated a general result, that is useful in a variety of situations, which enable us to transform the density function of a matrix $\mathbf{X} \in \mathfrak{P}_m^\beta$ to the density function of its eigenvalues, see Díaz-García and Gutiérrez-Jáimez (2009b).

Lemma 2.4. *Let $\mathbf{X} \in \mathfrak{P}_m^\beta$ be a random matrix with a density function $f_{\mathbf{X}}(\mathbf{X})$. Then the joint density function of the eigenvalues $\lambda_1, \dots, \lambda_m$ of $\mathbf{X}$ is*

$$\frac{\pi^{m^2\beta/2+\varrho}}{\Gamma_m^\beta[m\beta/2]}\prod_{i<j}^m(\lambda_i-\lambda_j)^\beta \int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} f(\mathbf{HLH}^*)(d\mathbf{H}) \qquad (18)$$

where $\mathbf{L} = \operatorname{diag}(\lambda_1, \dots, \lambda_m)$, $\lambda_1 > \cdots > \lambda_m > 0$, $(d\mathbf{H})$ *is the normalised Haar measure and*

$$\varrho = \begin{cases} 0, & \beta = 1; \\ -m, & \beta = 2; \\ -2m, & \beta = 4; \\ -4m, & \beta = 8. \end{cases}$$

Finally, let's recall the multidimensional convolution theorem in terms of the Laplace transform. For this purpose, let's use the complexification $\mathfrak{S}_m^{\beta,\mathfrak{C}} = \mathfrak{S}_m^\beta + i\mathfrak{S}_m^\beta$ of $\mathfrak{S}_m^\beta$. That is, $\mathfrak{S}_m^{\beta,\mathfrak{C}}$ consist of all matrices $\mathbf{X} \in (\mathfrak{F}^{\mathfrak{C}})^{m\times m}$ of the form $\mathbf{Z} = \mathbf{X} + i\mathbf{Y}$, with $\mathbf{X}, \mathbf{Y} \in \mathfrak{S}_m^\beta$. It comes to $\mathbf{X} = \operatorname{Re}(\mathbf{Z})$ and $\mathbf{Y} = \operatorname{Im}(\mathbf{Z})$ as the *real and imaginary parts* of $\mathbf{Z}$, respectively. The *generalised right half-plane* $\mathbf{\Phi}_m^\beta = \mathfrak{P}_m^\beta + i\mathfrak{S}_m^\beta$ in $\mathfrak{S}_m^{\beta,\mathfrak{C}}$ consists of all $\mathbf{Z} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$ such that $\operatorname{Re}(\mathbf{Z}) \in \mathfrak{P}_m^\beta$, see (Gross and Richards, 1987, p. 801).

Definition 2.1. If $f(\mathbf{X})$ is a function of $\mathbf{X} \in \mathfrak{P}_m^\beta$, the Laplace transform of $f(\mathbf{X})$ is defined to be

$$g(\mathbf{T}) = \int_{\mathbf{X}\in\mathfrak{P}_m^\beta} \operatorname{etr}\{-\mathbf{X}\mathbf{Z}\} f(\mathbf{X})(d\mathbf{X}). \tag{19}$$

where $\mathbf{Z} \in \mathbf{\Phi}_m^\beta$.

Lemma 2.5. *If* $g_1(\mathbf{Z})$ *and* $g_2(\mathbf{Z})$ *are the respective Laplace transforms of the densities* $f_{\mathbf{X}}(\mathbf{X})$ *and* $g_{\mathbf{Y}}(\mathbf{Y})$ *then the product* $g_1(\mathbf{Z})g_2(\mathbf{Z})$ *is the Laplace transform of the convolution* $f_{\mathbf{X}}(\mathbf{X}) * g_{\mathbf{Y}}(\mathbf{Y})$, *where*

$$h_{\mathbf{\Xi}}(\mathbf{\Xi}) = f_{\mathbf{X}}(\mathbf{X}) * g_{\mathbf{Y}}(\mathbf{Y}) = \int_{\mathbf{0}<\mathbf{\Upsilon}<\mathbf{\Xi}} f_{\mathbf{X}}(\mathbf{\Upsilon}) g_{\mathbf{Y}}(\mathbf{\Xi} - \mathbf{\Upsilon})(d\mathbf{\Upsilon}), \tag{20}$$

with $\mathbf{\Xi} = \mathbf{X} + \mathbf{Y}$ *and* $\mathbf{\Upsilon} = \mathbf{X}$.

3 Riesz distributions

This section shows two versions of the Riesz distributions (Díaz-García, 2012) and the study of their Bartlett decompositions. Also, inverse Riesz distributions are obtained.

Definition 3.1. Let $\mathbf{\Sigma} \in \mathbf{\Phi}_m^\beta$ and $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$.

1. Then it is said that $\mathbf{X}$ has a Riesz distribution of type I if its density function is

$$\frac{\beta^{am+\sum_{i=1}^m k_i}}{\Gamma_m^\beta[a,\kappa]|\mathbf{\Sigma}|^a q_\kappa(\mathbf{\Sigma})} \operatorname{etr}\{-\mathbf{\Sigma}^{-1}\mathbf{X}\}|\mathbf{X}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{X})(d\mathbf{X}) \tag{21}$$

   for $\operatorname{Re}(a) \geq (m-1)\beta/2 - k_m$; denoting this fact as $\mathbf{X} \sim \mathfrak{R}_m^{\beta,I}(a,\kappa,\mathbf{\Sigma})$.

2. Then it is said that $\mathbf{X}$ has a Riesz distribution of type II if its density function is

$$\frac{q_\kappa(\mathbf{\Sigma})\beta^{am-\sum_{i=1}^m k_i}}{\Gamma_m^\beta[a,-\kappa]|\mathbf{\Sigma}|^a} \operatorname{etr}\{-\mathbf{\Sigma}^{-1}\mathbf{X}\}|\mathbf{X}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{X}^{-1})(d\mathbf{X}) \tag{22}$$

   for $\operatorname{Re}(a) > (m-1)\beta/2 + k_1$; denoting this fact as $\mathbf{X} \sim \mathfrak{R}_m^{\beta,II}(a,\kappa,\mathbf{\Sigma})$.

Theorem 3.1. *Let* $\mathbf{\Sigma} \in \mathbf{\Phi}_m^\beta$ *and* $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$. *And let* $\mathbf{Y} = \mathbf{X}^{-1}$.

1. *Then if* $\mathbf{X}$ *has a Riesz distribution of type I the density of* $\mathbf{Y}$ *is*

$$\frac{\beta^{am+\sum_{i=1}^{m}k_i}}{\Gamma_m^\beta[a,\kappa]|\mathbf{\Sigma}|^a q_\kappa(\mathbf{\Sigma})}\operatorname{etr}\{-\mathbf{\Sigma}^{-1}\mathbf{Y}^{-1}\}|\mathbf{Y}|^{-(a+(m-1)\beta/2+1)}q_\kappa(\mathbf{Y}^{-1})(d\mathbf{Y}) \tag{23}$$

for $\operatorname{Re}(a) \geq (m-1)\beta/2 - k_m$ *and is termed as* inverse Riesz distribution of type I.

2. *Then if* $\mathbf{X}$ *has a Riesz distribution of type II the density of* $\mathbf{Y}$ *is*

$$\frac{q_\kappa(\mathbf{\Sigma})\beta^{am-\sum_{i=1}^{m}k_i}}{\Gamma_m^\beta[a,-\kappa]|\mathbf{\Sigma}|^a}\operatorname{etr}\{-\mathbf{\Sigma}^{-1}\mathbf{Y}^{-1}\}|\mathbf{Y}|^{-(a+(m-1)\beta/2+1)}q_\kappa(\mathbf{Y})(d\mathbf{Y}) \tag{24}$$

for $\operatorname{Re}(a) > (m-1)\beta/2 + k_1$ *and it said that* $\mathbf{Y}$ *has a* inverse Riesz distribution of type II.

Proof. It is immediately noted that $(d\mathbf{X}) = |\mathbf{Y}|^{-\beta(m-1)-2}(d\mathbf{Y})$ and from (21) and (22). □ □

Observe that, if $\kappa = (0, 0, \dots, 0)$ in two densities in Definition 3.1 and Theorem 3.1 the matrix multivariate gamma and inverse gamma distributions are obtained. As consequence, in this last case if $\mathbf{\Sigma} = 2\mathbf{\Sigma}$ and $a = \beta n/2$, the Wishart and inverse Wishart distributions are obtained, too.

Theorem 3.2. *Let* $\mathbf{T} \in \mathfrak{T}_U^\beta(m)$ *with* $t_{ii} > 0$, $i = 1, 2, \dots, m$ *and define* $\mathbf{X} = \mathbf{T}^*\mathbf{T}$.

1. *If* $\mathbf{X}$ *has a Riesz distribution of type I, (21), with* $\mathbf{\Sigma} = \mathbf{I}_m$, *then the elements* t_{ij} $(1 \leq i \leq j \leq m)$ *of* $\mathbf{T}$ *are all independent. Furthermore,* $t_{ii}^2 \sim \mathcal{G}^\beta(a + k_i - (i-1)\beta/2, 1)$ *and* $\sqrt{2}t_{ij} \sim \mathcal{N}_1^\beta(0,1)$ $(1 \leq i < j \leq m)$.

2. *If* $\mathbf{X}$ *has a Riesz distribution of type II, (22), with* $\mathbf{\Sigma} = \mathbf{I}_m$, *then the elements* t_{ij} $(1 \leq i \leq j \leq m)$ *of* $\mathbf{T}$ *are all independent. Moreover,* $t_{ii}^2 \sim \mathcal{G}^\beta(a - k_i - (i-1)\beta/2, 1)$ *and* $\sqrt{2}t_{ij} \sim \mathcal{N}_1^\beta(0,1)$ $(1 \leq i < j \leq m)$.

Where $x \sim \mathcal{G}^\beta(a, \alpha)$ *denotes a gamma distribution with parameters* a *and* α *and* $y \sim \mathcal{N}_1^\beta(0,1)$ *denotes a random variable with standard normal distribution for real normed division algebras. Moreover, their respective densities are*

$$\mathcal{G}^\beta(x : a, \alpha) = \frac{1}{(\alpha/\beta)^a\Gamma[a]}\exp\{-\beta x/\alpha\}x^{a-1}(dx),$$

and

$$\mathcal{N}_1^\beta(y : 0, 1) = \frac{1}{(2\pi/\beta)^{\beta/2}}\exp\{-\beta y^2/2\}(dy)$$

where $Re(a) > 0$ *and* $\alpha \in \mathbf{\Phi}_1^\beta$, *see Díaz-García and Gutiérrez-Jáimez (2011).*

Proof. This is given for the case of Riesz distribution type I. The proof for Riesz distribution type II is the same thing. The density of $\mathbf{X}$ is

$$\frac{\beta^{am+\sum_{i=1}^{m}k_i}}{\Gamma_m^\beta[a,\kappa]}\operatorname{etr}\{-\mathbf{X}\}|\mathbf{X}|^{a-(m-1)\beta/2-1}q_\kappa(\mathbf{X})(d\mathbf{X}). \tag{25}$$

Since $\mathbf{X} = \mathbf{T}^*\mathbf{T}$ we have

$$\begin{aligned}
\operatorname{tr}\mathbf{X} &= \operatorname{tr}\mathbf{T}^*\mathbf{T} = \sum_{i\leq j}^{m} t_{ij}^2, \\
|\mathbf{X}| &= |\mathbf{T}^*\mathbf{T}| = |\mathbf{T}|^2 = \prod_{i=1}^{m} t_{ii}^2, \\
q_\kappa(\mathbf{X}) &= q_\kappa(\mathbf{T}^*\mathbf{T}) = |\mathbf{T}^*\mathbf{T}|^{k_m} \prod_{i=1}^{m-1} |\mathbf{T}_i^*\mathbf{T}_i|^{k_i - k_{i+1}} = \prod_{i=1}^{m} t_{ii}^{2k_i},
\end{aligned}$$

and by Theorem 2.2 noting that $dt_{ii}^2 = 2t_{ii}dt_{ii}$, then

$$\begin{aligned}
(d\mathbf{X}) &= 2^m \prod_{i=1}^{m} t_{ii}^{\beta(m-i)+1} \left(\bigwedge_{i\leq j} dt_{ij}\right), \\
&= \prod_{i=1}^{m} \left(t_{ii}^2\right)^{\beta(m-i)/2} \left(\bigwedge_{i=1} dt_{ii}^2\right) \wedge \left(\bigwedge_{i<j} dt_{ij}\right).
\end{aligned}$$

Substituting this expression in (25) and using (5) we find that the joint density of the t_{ij} $(1 \leq i \leq j \leq m)$ can be written as

$$\prod_{i=1}^{m} \frac{\beta^{a+k_i-(i-1)\beta/2}}{\Gamma[a+k_i-(i-1)\beta/2]} \exp\{-\beta t_{ii}^2\} \left(t_{ii}^2\right)^{a+k_i-(i-1)\beta/2-1} \left(dt_{ii}^2\right)$$

$$\times \prod_{i<j}^{m} \frac{1}{(\pi/\beta)^{\beta/2}} \exp\{-\beta t_{ij}^2\}(dt_{ij}),$$

only observe that

$$\begin{aligned}
\frac{\beta^{am+\sum_{i=1}^m k_i}}{\Gamma_m^\beta[a,\kappa]} &= \frac{\beta^{am+\sum_{i=1}^m k_i - m(m-1)\beta/4}}{\beta^{-m(m-1)\beta/4}\pi^{m(m-1)\beta/4}\prod_{i=1}^m \Gamma[a+k_i-(i-1)\beta/2]} \\
&= \prod_{i=1}^{m} \frac{\beta^{a+k_i-(i-1)\beta/2}}{\Gamma[a+k_i-(i-1)\beta/2]} \prod_{i<j}^{m} \frac{1}{(\pi/\beta)^{\beta/2}}. \quad \square
\end{aligned}$$

$\square$

In analogy to generalised variance for Wishart case, the following result gives the distribution of $|\mathbf{X}|$ when $\mathbf{X}$ has a Riesz distribution type I or type II.

Theorem 3.3. *Let $v = |\mathbf{X}|/|\mathbf{\Sigma}|$. Then*

1. *if $\mathbf{X}$ has a Riesz distribution of type I, (21), the density of v is*

$$\prod_{i=1}^{m} \mathcal{G}^\beta\left(t_{ii}^2 : a+k_i-(i-1)\beta/2, 1\right).$$

2. *if $\mathbf{X}$ has a Riesz distribution of type II, (22), the density of v is*

$$\prod_{i=1}^{m} \mathcal{G}^\beta\left(t_{ii}^2 : a-k_i-(i-1)\beta/2, 1\right).$$

where t_{ii}^2, $i = 1, \dots, m$, are independent random variables.

Proof. This is immediately from Theorem 3.2, noting that if

$$\mathbf{B} = \mathbf{\Sigma}^{-1/2}\mathbf{X}\mathbf{\Sigma}^{-1/2} = \mathbf{T}^*\mathbf{T},$$

with $\mathbf{T} \in \mathfrak{T}_U^\beta(m)$ and $t_{ii} > 0$, $i = 1, 2, \dots, m$, then

$$|\mathbf{B}| = \prod_{i=1}^m t_{ii}^2 = |\mathbf{X}|/|\mathbf{\Sigma}| = v. \quad \square$$

$\square$

4 Generalised beta distributions: Beta-Riesz distributions.

This section defines several versions for the beta functions and their relation with the gamma functions type I and II. In these terms, the beta-Riesz distributions type I and II are defined. Finally, diverse properties are studied.

4.1 Generalised c-beta function

A generalised of *multivariate beta function* for the cone $\mathfrak{P}_m^\beta$, denoted as $\mathcal{B}_m^\beta[a, \kappa; b, \tau]$, can be defined as

$$\int_{\mathbf{0}<\mathbf{S}<\mathbf{I}_m} |\mathbf{S}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{S}) |\mathbf{I}_m - \mathbf{S}|^{b-(m-1)\beta/2-1} q_\tau(\mathbf{I}_m - \mathbf{S})(d\mathbf{S}) \tag{26}$$

where $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$, $\tau = (t_1, t_2, \dots, t_m)$, $t_1 \geq t_2 \geq \cdots \geq t_m \geq 0$, $\mathrm{Re}(a) > (m-1)\beta/2 - k_m$ and $\mathrm{Re}(b) > (m-1)\beta/2 - t_m$. This is defined by Faraut and Korányi (1994, p. 130) for Euclidean simple Jordan algebras. In the context of multivariate analysis, this generalised beta function can be termed *generalised c-beta function type I*, as analogy to the correspondence case of matrix multivariate beta distribution, and using the term c-beta as abbreviation of classical-beta. In the next theorem we introduce the *generalised c-beta function type II* and its relation with the generalised gamma function.

Theorem 4.1. *The generalised c-beta function type I can be expressed as*

$$\int_{\mathbf{R}\in\mathfrak{P}_m^\beta} |\mathbf{R}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{R}) |\mathbf{I}_m + \mathbf{R}|^{-(a+b)} q_{\kappa+\tau}^{-1}(\mathbf{I}_m + \mathbf{R})(d\mathbf{R})$$

$$= \frac{\Gamma_m^\beta[a, \kappa]\Gamma_m^\beta[b, \tau]}{\Gamma_m^\beta[a + b, \kappa + \tau]},$$

where $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$, $\tau = (t_1, t_2, \dots, t_m)$, $t_1 \geq t_2 \geq \cdots \geq t_m \geq 0$, Re$(a) > (m-1)\beta/2 - k_m$ and Re$(b) > (m-1)\beta/2 - t_m$. The integral expression is termed generalised c-beta function type II.

Proof. Let $\mathbf{R} = (\mathbf{I} - \mathbf{S})^{-1} - \mathbf{I}$ then by Lemma 2.3 $(d\mathbf{S}) = (\mathbf{I} + \mathbf{R})^{-\beta(m-1)-2}(d(\mathbf{R}))$. The desired result is obtained making the change of variable on (26) noting that $\mathbf{S} = (\mathbf{I}_m + \mathbf{R})^{-1/2}\mathbf{R}(\mathbf{I}_m + \mathbf{R})^{-1/2}$, then

$$|\mathbf{S}| = |\mathbf{R}||\mathbf{I}_m + \mathbf{R}|^{-1},$$

$$|\mathbf{I}_m - \mathbf{S}| = |\mathbf{I}_m + \mathbf{R}|^{-1},$$

and by (6) and (11) we have

$$q_\kappa((\mathbf{I}_m+\mathbf{R})^{-1/2}\mathbf{R}(\mathbf{I}_m+\mathbf{R})^{-1/2})q_\tau((\mathbf{I}_m+\mathbf{R})^{-1}) = q_\kappa(\mathbf{R})q_{\kappa+\tau}^{-1}(\mathbf{I}_m+\mathbf{R}).$$

For the expression in terms of generalised gamma function, let $\mathbf{B}=\mathbf{\Xi}^{1/2}\mathbf{S}\mathbf{\Xi}^{1/2}$ in (26), such that $(\mathbf{\Xi}^{1/2})^2=\mathbf{\Xi}$. Then $(d\mathbf{S}) = |\mathbf{\Xi}|^{-(m-1)\beta/2-1}(d\mathbf{B})$, and

$$\begin{aligned}\mathcal{B}_m^\beta[a,\kappa;b,\tau]|\mathbf{\Xi}|^{a+b-(m-1)\beta/2-1}q_{\kappa+\tau}(\mathbf{\Xi})\\ = \int_{\mathbf{0}}^{\mathbf{\Xi}}|\mathbf{B}|^{a-(m-1)\beta/2-1}q_\kappa(\mathbf{B})|\mathbf{\Xi}-\mathbf{B}|^{b-(m-1)\beta/2-1}q_\tau(\mathbf{\Xi}-\mathbf{B})(d\mathbf{B}).\end{aligned}$$

Taking Laplace transform of both size, by (21), the left size is

$$\begin{aligned}\int_{\mathbf{\Xi}\in\mathfrak{P}_m^\beta}\mathcal{B}_m^\beta[a,\kappa;b,\tau]\operatorname{etr}\{-\mathbf{\Xi}\mathbf{Z}\}|\mathbf{\Xi}|^{a+b-(m-1)\beta/2-1}q_{\kappa+\tau}(\mathbf{\Xi})(d\mathbf{\Xi})\\ = \mathcal{B}_m^\beta[a,\kappa;b,\tau]\Gamma_m^\beta[a+b;\kappa+\tau]|\mathbf{Z}|^{-(a+b)}q_{\kappa+\tau}(\mathbf{Z}),\end{aligned}$$

and applying Lemma 2.5, $g_1(\mathbf{Z})$ is

$$\int_{\mathbf{\Xi}\in\mathfrak{P}_m^\beta}\operatorname{etr}\{-\mathbf{\Xi}\mathbf{Z}\}|\mathbf{\Xi}|^{a-(m-1)\beta/2-1}q_\kappa(\mathbf{\Xi})(d\mathbf{\Xi}) = \Gamma_m^\beta[a;\kappa]|\mathbf{Z}|^{-a}q_\kappa(\mathbf{Z}),$$

and $g_2(\mathbf{Z})$ is given by

$$\int_{\mathbf{\Xi}\in\mathfrak{P}_m^\beta}\operatorname{etr}\{-\mathbf{\Xi}\mathbf{Z}\}|\mathbf{\Xi}-\mathbf{B}|^{b-(m-1)\beta/2-1}q_\kappa(\mathbf{\Xi}-\mathbf{B})(d\mathbf{\Xi}) = \Gamma_m^\beta[b;\tau]|\mathbf{Z}|^{-b}q_\tau(\mathbf{Z}).$$

Thus, equally

$$\mathcal{B}_m^\beta[a,\kappa;b,\tau] = \frac{\Gamma_m^\beta[a,\kappa]\Gamma_m^\beta[b,\tau]}{\Gamma_m^\beta[a+b,\kappa+\tau]}. \quad \square$$

$\square$

4.2 Generalised k-beta function

Alternatively, a generalised of *multivariate beta function* for the cone $\mathfrak{P}_m^\beta$, denoted as $\mathcal{B}_m^\beta[a,-\kappa;b,-\tau]$, can be defined as

$$\int_{\mathbf{0}<\mathbf{S}<\mathbf{I}_m}|\mathbf{S}|^{a-(m-1)\beta/2-1}q_\kappa(\mathbf{S}^{-1})|\mathbf{I}_m-\mathbf{S}|^{b-(m-1)\beta/2-1}q_\tau((\mathbf{I}_m-\mathbf{S})^{-1})(d\mathbf{S}) \qquad (27)$$

where $\kappa=(k_1,k_2,\dots,k_m)$, $k_1\geq k_2\geq\cdots\geq k_m\geq 0$, $\tau=(t_1,t_2,\dots,t_m)$, $t_1\geq t_2\geq\cdots\geq t_m\geq 0$, $\operatorname{Re}(a)>(m-1)\beta/2+k_1$ and $\operatorname{Re}(b)>(m-1)\beta/2+t_1$. Again, in the context of multivariate analysis, this generalised k-beta function can be termed *generalised k-beta function type I*, as an analogy to the corresponding case of matrix multivariate beta distribution and using the term k-beta as abbreviation of Khatri-beta. Next theorem introduces the *generalised k-beta function type II* and its relation with the generalised gamma function proposed by Khatri (1966).

Theorem 4.2. *The generalised k-beta function type II can be expressed as*

$$\int_{\mathbf{R}\in\mathfrak{P}_m^\beta}|\mathbf{R}|^{a-(m-1)\beta/2-1}q_\kappa^{-1}(\mathbf{R})|\mathbf{I}_m+\mathbf{R}|^{-(a+b)}q_{\kappa+\tau}(\mathbf{I}_m+\mathbf{R})(d\mathbf{R})$$

$$= \frac{\Gamma_m^\beta[a,-\kappa]\Gamma_m^\beta[b,-\tau]}{\Gamma_m^\beta[a+b,-\kappa-\tau]},$$

where $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$, $\tau = (t_1, t_2, \dots, t_m)$, $t_1 \geq t_2 \geq \cdots \geq t_m \geq 0$, *Re*$(a) > (m-1)\beta/2 + k_1$ *and Re*$(b) > (m-1)\beta/2 + t_1$. *The integral expression is termed* generalised k-beta function type II.

Proof. The proof is analogous to the given for Theorem 4.1. □ □

Observe that if $\kappa = (0, \dots, 0)$ and $\tau = (0, \dots, 0)$ in (26), Theorem 4.1, (27) and Theorem 4.2 the beta function (14) is obtained.

4.3 c-beta-Riesz and k-beta-Riesz distributions

As an immediate consequence of the results of the previous section, next the c-beta-Riesz and k-beta-Riesz distributions types I and II are defined.

Definition 4.1. Let $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$ and $\tau = (t_1, t_2, \dots, t_m)$, $t_1 \geq t_2 \geq \cdots \geq t_m \geq 0$.

1. Then it said that $\mathbf{S}$ has a *c-beta-Riesz distribution of type I* if its density function is

$$\frac{1}{\mathcal{B}_m^\beta[a,\kappa;b,\tau]}|\mathbf{S}|^{a-(m-1)\beta/2-1}q_\kappa(\mathbf{S})|\mathbf{I}_m-\mathbf{S}|^{b-(m-1)\beta/2-1}q_\tau(\mathbf{I}_m-\mathbf{S})(d\mathbf{S}), \qquad (28)$$

   where $\mathbf{0} < \mathbf{S} < \mathbf{I}_m$ and *Re*$(a) > (m-1)\beta/2 - k_m$ and $\text{Re}(b) > (m-1)\beta/2 - t_m$.

2. Then it said that $\mathbf{R}$ has a *c-beta-Riesz distribution of type II* if its density function is

$$\frac{1}{\mathcal{B}_m^\beta[a,\kappa;b,\tau]}|\mathbf{R}|^{a-(m-1)\beta/2-1}q_\kappa(\mathbf{R})|\mathbf{I}_m+\mathbf{R}|^{-(a+b)}q_{\kappa+\tau}^{-1}(\mathbf{I}_m+\mathbf{R})(d\mathbf{R}), \qquad (29)$$

   where $\mathbf{R} \in \mathfrak{P}_m^\beta$ and $\text{Re}(a) > (m-1)\beta/2 - k_m$ and $\text{Re}(b) > (m-1)\beta/2 - t_m$.

Similarly we have

Definition 4.2. Let $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$ and $\tau = (t_1, t_2, \dots, t_m)$, $t_1 \geq t_2 \geq \cdots \geq t_m \geq 0$.

1. Then it said that $\mathbf{S}$ has a *k-beta-Riesz distribution of type I* if its density function is

$$\frac{1}{\mathcal{B}_m^\beta[a,-\kappa;b,-\tau]}|\mathbf{S}|^{a-(m-1)\beta/2-1}q_\kappa^{-1}(\mathbf{S})|\mathbf{I}_m-\mathbf{S}|^{b-(m-1)\beta/2-1}q_\tau^{-1}(\mathbf{I}_m-\mathbf{S})(d\mathbf{S}), \quad (30)$$

   where $\mathbf{0} < \mathbf{S} < \mathbf{I}_m$ and $\text{Re}(a) > (m-1)\beta/2 + k_1$ and $\text{Re}(b) > (m-1)\beta/2 + t_1$.

2. Then it said that $\mathbf{R}$ has a *k-beta-Riesz distribution of type II* if its density function is

$$\frac{1}{\mathcal{B}_m^\beta[a,-\kappa;b,-\tau]}|\mathbf{R}|^{a-(m-1)\beta/2-1}q_\kappa^{-1}(\mathbf{R})|\mathbf{I}_m+\mathbf{R}|^{-(a+b)}q_{\kappa+\tau}(\mathbf{I}_m+\mathbf{R})(d\mathbf{R}), \qquad (31)$$

   where $\mathbf{R} \in \mathfrak{P}_m^\beta$ and $\text{Re}(a) > (m-1)\beta/2 + k_1$ and $\text{Re}(b) > (m-1)\beta/2 + t_1$.

Observe that the relationship between the densities (28) and (29), and between the densities (30) and (31) are easily obtained from the theorems 4.1 and 4.2, respectively.

The following result state the relation between the Riesz and beta-Riesz distributions.

Theorem 4.3. *Let* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *be independently distributed as Riesz distribution type I, such that* $\mathbf{X}_1 \sim \mathfrak{R}_m^{\beta,I}(a,\kappa,\mathbf{I}_m)$ *and* $\mathbf{X}_2 \sim \mathfrak{R}_m^{\beta,I}(b,\tau,\mathbf{I}_m)$*,* $Re(a) > (m-1)\beta/2 + k_1$ *and* $Re(b) > (m-1)\beta/2 + t_1$*. Let*

$$\mathbf{S} = (\mathbf{X}_1+\mathbf{X}_2)^{-1/2}\mathbf{X}_2(\mathbf{X}_1+\mathbf{X}_2)^{-1/2'},$$

where $(\mathbf{X}_1+\mathbf{X}_2)^{1/2}$ *is any nonsingular factorisation of* $(\mathbf{X}_1+\mathbf{X}_2)$*,* $(\mathbf{X}_1+\mathbf{X}_2) = (\mathbf{X}_1+\mathbf{X}_2)^{1/2}(\mathbf{X}_1+\mathbf{X}_2)^{1/2'}$*. Then* $\mathbf{S}$ *has a c-beta-Riesz distribution type I.*

Proof. The joint density of $\mathbf{X}_1$ and $\mathbf{X}_2$) is given by

$$\frac{\beta^{(a+b)m+\sum_{i=1}^m(k_i+t_i)}}{\Gamma_m^\beta[a,\kappa]\Gamma_m^\beta[b,\tau]}\operatorname{etr}\{-(\mathbf{X}_1+\mathbf{X}_2)\}|\mathbf{X}_1|^{a-(m-1)\beta/2-1}q_\kappa(\mathbf{X}_1)$$

$$\times|\mathbf{X}_2|^{b-(m-1)\beta/2-1}q_\tau(\mathbf{X}_2)(d\mathbf{X}_1)\wedge(d\mathbf{X}_2).$$

Let $\mathbf{Y} = \mathbf{X}_1+\mathbf{X}_2$ and $\mathbf{Z} = \mathbf{X}_2$, then, $(d\mathbf{X}_1)\wedge(d\mathbf{X}_2) = (d\mathbf{Y})\wedge(d\mathbf{Z})$. Then the joint density of $\mathbf{Y}$ and $\mathbf{Z}$ is given by

$$\frac{\beta^{(a+b)m+\sum_{i=1}^m(k_i+t_i)}}{\Gamma_m^\beta[a,\kappa]\Gamma_m^\beta[b,\tau]}\operatorname{etr}\{-\mathbf{Y}\}|\mathbf{Y}-\mathbf{Z}|^{a-(m-1)\beta/2-1}q_\kappa(\mathbf{Y}-\mathbf{Z})$$

$$\times|\mathbf{Z}|^{b-(m-1)\beta/2-1}q_\tau(\mathbf{Z})(d\mathbf{Y})\wedge(d\mathbf{Z}).$$

Let $\mathbf{S} = \mathbf{Y}^{-1/2}\mathbf{Z}\mathbf{Y}^{-1/2'}$ and $\mathbf{W} = \mathbf{Y}$, with $\mathbf{Y} = \mathbf{Y}^{1/2'}\mathbf{Y}^{1/2}$, then,

$$(d\mathbf{Y})\wedge(d\mathbf{Z}) = |\mathbf{W}|^{\beta(m-1)/2+1}(d\mathbf{W})\wedge(d\mathbf{S})$$

Hence the joint density of $\mathbf{S}$ and $\mathbf{W}$ is

$$\frac{\beta^{(a+b)m+\sum_{i=1}^m(k_i+t_i)}}{\Gamma_m^\beta[a,\kappa]\Gamma_m^\beta[b,\tau]}\operatorname{etr}\{-\mathbf{W}\}|\mathbf{W}|^{a+b-\beta(m-1)/2-1}q_{\kappa+\tau}(\mathbf{W})|\mathbf{I}-\mathbf{S}|^{a-(m-1)\beta/2-1}$$

$$\times q_\kappa(\mathbf{I}-\mathbf{S})|\mathbf{S}|^{b-(m-1)\beta/2-1}q_\tau(\mathbf{S})(d\mathbf{W})\wedge(d\mathbf{S}).$$

Therefore, integrating with respect to $\mathbf{W}$ the desired result is obtained. □ □

Theorem 4.4. *Let* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *be independently distributed as Riesz distribution type I, such that* $\mathbf{X}_1 \sim \mathfrak{R}_m^{\beta,I}(a,\kappa,\mathbf{I}_m)$ *and* $\mathbf{X}_2 \sim \mathfrak{R}_m^{\beta,I}(b,\tau,\mathbf{I}_m)$*,* $Re(a) > (m-1)\beta/2 + k_1$ *and* $Re(b) > (m-1)\beta/2 + t_1$*. Let*

$$\mathbf{R} = \mathbf{X}_1^{-1/2}\mathbf{X}_2\mathbf{X}_1^{-1/2'},$$

where $\mathbf{X}_1^{1/2}$ *is any nonsingular factorisation of* $\mathbf{X}_1$*, in the sense that* $\mathbf{X}_1 = \mathbf{X}_1^{1/2'}\mathbf{X}_1^{1/2}$*. Then* $\mathbf{S}$ *has a c-beta-Riesz distribution type II.*

Proof. The joint density of $\mathbf{X}_1$ and $\mathbf{X}_2$) is given by

$$\frac{\beta^{(a+b)m+\sum_{i=1}^m(k_i+t_i)}}{\Gamma_m^\beta[a,\kappa]\Gamma_m^\beta[b,\tau]}\operatorname{etr}\{-(\mathbf{X}_1+\mathbf{X}_2)\}|\mathbf{X}_1|^{a-(m-1)\beta/2-1}q_\kappa(\mathbf{X}_1)$$

$$\times|\mathbf{X}_2|^{b-(m-1)\beta/2-1}q_\tau(\mathbf{X}_2)(d\mathbf{X}_1)\wedge(d\mathbf{X}_2).$$

Let $\mathbf{R} = \mathbf{X}_1^{-1/2}\mathbf{X}_2\mathbf{X}_1^{-1/2'}$ and $\mathbf{W} = \mathbf{X}_1$, with $\mathbf{X}_1 = \mathbf{X}_1^{1/2'}\mathbf{X}_1^{1/2}$, then,

$$(d\mathbf{X}_1)\wedge(d\mathbf{X}_2) = |\mathbf{W}|^{\beta(m-1)/2+1}(d\mathbf{W})\wedge(d\mathbf{R})$$

Hence the joint density of $\mathbf{S}$ and $\mathbf{W}$ is

$$\frac{\beta^{(a+b)m+\sum_{i=1}^{m}(k_i+t_i)}}{\Gamma_m^\beta[a,\kappa]\Gamma_m^\beta[b,\tau]}\operatorname{etr}\{-\mathbf{W}(\mathbf{I}_m+\mathbf{R})\}|\mathbf{W}|^{a+b-(m-1)\beta/2-1}q_{\kappa+\tau}(\mathbf{W})$$

$$\times|\mathbf{R}|^{b-(m-1)\beta/2-1}q_\tau(\mathbf{R})(d\mathbf{W})\wedge(d\mathbf{R}).$$

Then, integrating with respect to $\mathbf{W}$ the density of $\mathbf{R}$ is obtained. $\square$ $\square$

The following theorems 4.5 and 4.6 contain versions for k-beta-Riesz distributions of theorems 4.3 and 4.4, whose proofs are similar.

Theorem 4.5. *Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be independently distributed as Riesz distribution type II, such that $\mathbf{X}_1 \sim \mathfrak{R}_m^{\beta,II}(a,\kappa,\mathbf{I}_m)$ and $\mathbf{X}_2 \sim \mathfrak{R}_m^{\beta,II}(b,\tau,\mathbf{I}_m)$, $Re(a) > (m-1)\beta/2+k_1$ and $Re(b) > (m-1)\beta/2+t_1$. Let*

$$\mathbf{S} = (\mathbf{X}_1+\mathbf{X}_2)^{-1/2}\mathbf{X}_2(\mathbf{X}_1+\mathbf{X}_2)^{-1/2'},$$

where $(\mathbf{X}_1+\mathbf{X}_2)^{1/2}$ is any nonsingular factorisation of $(\mathbf{X}_1+\mathbf{X}_2)$, $(\mathbf{X}_1+\mathbf{X}_2) = (\mathbf{X}_1+\mathbf{X}_2)^{1/2}(\mathbf{X}_1+\mathbf{X}_2)^{1/2'}$. Then $\mathbf{S}$ has a k-beta-Riesz distribution type I.

Theorem 4.6. *Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be independently distributed as Riesz distribution type I, such that $\mathbf{X}_1 \sim \mathfrak{R}_m^{\beta,II}(a,\kappa,\mathbf{I}_m)$ and $\mathbf{X}_2 \sim \mathfrak{R}_m^{\beta,II}(b,\tau,\mathbf{I}_m)$, $Re(a) > (m-1)\beta/2+k_1$ and $Re(b) > (m-1)\beta/2+t_1$. Let*

$$\mathbf{R} = \mathbf{X}_1^{-1/2}\mathbf{X}_2\mathbf{X}_1^{-1/2'},$$

where $\mathbf{X}_1^{1/2}$ is any nonsingular factorisation of $\mathbf{X}_1$, in the sense $\mathbf{X}_1 = \mathbf{X}_1^{1/2'}\mathbf{X}_1^{1/2}$. Then $\mathbf{S}$ has a k-beta-Riesz distribution type II.

4.4 Some properties of the c-beta-Riesz and k-beta-Riesz distributions

This section derives the distributions of eigenvalues for c-beta-Riesz and k-beta-Riesz distributions type I and II. First remember that:

Remark 4.1. If $\operatorname{ch}_i(\mathbf{M})$ denotes the i-th eigenvalue of the matrix $\mathbf{M} \in \mathfrak{P}_m^\beta$, then note that if $\mathbf{X},\mathbf{Y} \in \mathfrak{P}_m^\beta$,

$$\operatorname{ch}_i(\mathbf{X}^{-1/2}\mathbf{Y}\mathbf{X}^{-1/2}) = \operatorname{ch}_i(\mathbf{X}^{-1}\mathbf{Y}) = \operatorname{ch}_i(\mathbf{Y}\mathbf{X}^{-1}).$$

Furthermore, if $\mathbf{A} \in \mathbf{\Phi}_m^\beta$ is a nonsingular matrix then

$$\operatorname{ch}_i((\mathbf{A}'\mathbf{X}\mathbf{A})^{-1}\mathbf{A}'\mathbf{Y}\mathbf{A}) = \operatorname{ch}_i(\mathbf{X}^{-1}\mathbf{Y})$$

Then, it is found the joint distributions of the eigenvalues for random matrices c- and k-beta-Riesz type I and II assuming that $\mathbf{\Sigma} \neq \mathbf{I}_m$ in theorems 4.3-4.6.

Theorem 4.7. *Let $\mathbf{\Sigma} \in \mathbf{\Phi}_m^\beta$, $\kappa = (k_1,k_2,\dots,k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$ and $\tau = (t_1,t_2,\dots,t_m)$, $t_1 \geq t_2 \geq \cdots \geq t_m \geq 0$.*

1. *Let $\lambda_1,\dots,\lambda_m$, $\lambda_1 > \cdots > \lambda_m > 0$ be the eigenvalues of $\mathbf{S}$. Then if $\mathbf{S}$ has a c-beta-Riesz distribution of type I, the joint density of $\lambda_1,\dots,\lambda_m$ is*

$$\frac{\pi^{m^2\beta/2+\varrho}}{\Gamma_m^\beta[m\beta/2]\mathcal{B}_m^\beta[a,\kappa;b,\tau]}\prod_{i<j}^{m}(\lambda_i-\lambda_j)^\beta\prod_{i=1}^{m}\lambda_i^{a+k_i-(m-1)\beta/2-1}$$

$$\prod_{i=1}^{m}(1-\lambda_i)^{b+t_i-(m-1)\beta/2-1}\left(\bigwedge_{i=1}^{m} d\lambda_i\right),$$

where $0 < \lambda_i < 1$, $i = 1,\dots,m$ *and* $Re(a) > (m-1)\beta/2 - k_m$ *and* $Re(b) > (m-1)\beta/2 - t_m$.

2. *Let* $\delta_1,\dots,\delta_m$, $\delta_1 > \cdots > \delta_m > 0$ *be the eigenvalues of* $\mathbf{R}$. *Then if* $\mathbf{R}$ *has a c-beta-Riesz distribution of type II, the joint density of their eigenvalues is*

$$\frac{\pi^{m^2\beta/2+\varrho}}{\Gamma_m^\beta[m\beta/2]\mathcal{B}_m^\beta[a,\kappa;b,\tau]}\prod_{i<j}^{m}(\delta_i-\delta_j)^\beta\prod_{i=1}^{m}\delta_i^{a+k_i-(m-1)\beta/2-1}$$

$$\prod_{i=1}^{m}(1-\delta_i)^{-(a+b+k_i+t_i)}\left(\bigwedge_{i=1}^{m} d\delta_i\right),$$

where $\delta_i > 0$, $i = 1,\dots,m$ *and* $Re(a) > (m-1)\beta/2 - k_m$ *and* $Re(b) > (m-1)\beta/2 - t_m$.

ϱ *is defined in Lemma 2.4.*

Proof. This is due to applying the Lemma 2.4 in (28) and (29), and taking into account the Remark 4.1 and equations (5) and (6). □ □

This section conclude establishing the Theorem 4.7 for the case of the k-beta-Riesz distributions.

Theorem 4.8. *Let* $\mathbf{\Sigma} \in \mathbf{\Phi}_m^\beta$, $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$ *and* $\tau = (t_1, t_2, \dots, t_m)$, $t_1 \geq t_2 \geq \cdots \geq t_m \geq 0$.

1. *Let* $\lambda_1,\dots,\lambda_m$, $\lambda_1 > \cdots > \lambda_m > 0$ *be the eigenvalues of* $\mathbf{S}$. *Then if* $\mathbf{S}$ *has a k-beta-Riesz distribution of type I, the joint density of* $\lambda_1,\dots,\lambda_m$ *is*

$$\frac{\pi^{m^2\beta/2+\varrho}}{\Gamma_m^\beta[m\beta/2]\mathcal{B}_m^\beta[a,-\kappa;b,-\tau]}\prod_{i<j}^{m}(\lambda_i-\lambda_j)^\beta\prod_{i=1}^{m}\lambda_i^{a-k_i-(m-1)\beta/2-1}$$

$$\prod_{i=1}^{m}(1-\lambda_i)^{b-t_i-(m-1)\beta/2-1}\left(\bigwedge_{i=1}^{m} d\lambda_i\right),$$

where $0 < \lambda_i < 1$, $i = 1,\dots,m$ *and* $Re(a) > (m-1)\beta/2 + k_1$ *and* $Re(b) > (m-1)\beta/2 + t_1$.

2. *Let* $\delta_1,\dots,\delta_m$, $\delta_1 > \cdots > \delta_m > 0$ *be the eigenvalues of* $\mathbf{R}$. *Then if* $\mathbf{R}$ *has a k-beta-Riesz distribution of type II, the joint density of their eigenvalues is*

$$\frac{\pi^{m^2\beta/2+\varrho}}{\Gamma_m^\beta[m\beta/2]\mathcal{B}_m^\beta[a,-\kappa;b,-\tau]}\prod_{i<j}^{m}(\delta_i-\delta_j)^\beta\prod_{i=1}^{m}\delta_i^{a-k_i-(m-1)\beta/2-1}$$

$$\prod_{i=1}^{m}(1-\delta_i)^{-(a+b-k_i-t_i)}\left(\bigwedge_{i=1}^{m} d\delta_i\right),$$

where $\delta_i > 0$, $i = 1,\dots,m$ *and* $Re(a) > (m-1)\beta/2 + k_1$ *and* $Re(b) > (m-1)\beta/2 + t_1$.

ϱ *is defined in Lemma 2.4.*

Finally observe that if in all result of this section are taking $\kappa = (0,0,\dots,0)$ and $\tau = (0,0,\dots,0)$ the obtained results are the corresponding to matrix multivariate beta distributions of type I and II.

Conclusions

Finally, note that the real dimension of real normed division algebras can be expressed as powers of 2, $\beta = 2^n$ for $n = 0, 1, 2, 3$. On the other hand, as observed from Kabe (1984), the results obtained in this work can be extended to hypercomplex cases; that is, for complex, bicomplex, biquaternion and bioctonion (or sedenionic) algebras, which of course are not division algebras (except the complex algebra). Also note, that hypercomplex algebras are obtained by replacing the real numbers with complex numbers in the construction of real normed division algebras. Thus, the results for hypercomplex algebras are obtained by simply replacing β with 2β in our results. Alternatively, following Kabe (1984), it can be concluded that, results are true for '2^n-ions', $n = 0, 1, 2, 3, 4, 5$, emphasising that only for $n = 0, 1, 2, 3$ are the result algebras, in fact, real normed division algebras.

Acknowledgements

References

Baez, J. C. (2002). The octonions. *Bull. Amer. Math. Soc.*, Vol. 39, pp. 145–205.

Díaz-García, J. A., and Gutiérrez-Jáimez, R. (2009b). Special functions: Integral properties of Jack polynomials, hypergeometric functions and Invariant polynomials. `http://arxiv.org/abs/0909.1988`. Also submited.

Díaz-García, J. A., and Gutiérrez-Jáimez, R. (2011). On Wishart distribution: Some extensions. *Linear Algebra Appl.*, Vol. 435, pp. 1296-1310.

Díaz-García, J. A. (2012). Distributions on symmetric cones I: Riesz distribution. `http://arxiv.org/abs/1211.1746`. Also submited.

Dumitriu, I. (2002). *Eigenvalue statistics for beta-ensembles*. PhD thesis, Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA.

Edelman, A., and Rao, R. R. (2005). Random matrix theory. *Acta Numer.*, Vol. 14, pp. 233–297.

Ebbinghaus, H. D., Hermes, H. Hirzebruch, F., Koecher, M., Mainzer, K., Neukirch, J., Prestel, A., and Remmert, R. (1990). *Numbers.* GTM/RIM 123, H.L.S. Orde, tr., Springer, New York.

Faraut, J., and Korányi, A. (1994). *Analysis on symmetric cones.* Oxford Mathematical Monographs, Clarendon Press, Oxford.

Gross, K. I., and Richards, D. ST. P. (1987). Special functions of matrix argument I: Algebraic induction zonal polynomials and hypergeometric functions. *Trans. Amer. Math. Soc.*, Vol. 301, No. 2), pp. 475–501.

Hassairi, A., and Lajmi, S. (2001). Riesz exponential families on symmetric cones. *J. Theoret. Probab.*, Vol. 14, pp. 927-948.

Hassairi, A., Lajmi, S. and Zine, R. (2005) Beta-Riesz distributions on symmetric cones, *J. Statist. Plann. Inf.*, Vol. 133, pp. 387-404.

Herz, C. S. (1955). Bessel functions of matrix argument. *Ann. of Math.*, Vol. 61, No. 3, pp. 474-523.

Kabe, D. G. (1984). Classical statistical analysis based on a certain hypercomplex multivariate normal distribution. *Metrika*, Vol. 31, pp. 63–76.

Khatri, C. G. (1966) On certain distribution problems based on positive definite quadratic functions in normal vector. *Ann. Math. Statist.*, Vol. 37, pp. 468–479.